Choleski Algorithms for the Solution of a Set of Normal Equations

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Abstract:

Algorithms for the solution of a set of normal equations by the method of Choleski are developed. The algorithms are not new to the geomatic engineering profession: we have been using them since the 1960s. This author first used the algorithms as the basis of a horizontal control least squares adjustment program that he developed while attending the University of New Brunswick as a graduate student in 1973 - 1974.

Introduction:

A positive definite normal equation matrix can be transformed into a product of a lower triangular matrix and its transpose. The method used to find ***L*** and its transpose is well known as Choleski Decomposition, attributed to a Polish Mathematician. After computing ***L*** the "lower square root" of ***N***, it is a trivial matter to compute the vector of unknowns. Algorithms for the computation of the Variance-Covariance ***S*** matrix, the inverse of the normal equation matrix ***N***, will be derived in a second paper in this series. A third paper will derive Block Choleski algorithms for solving a vast set of normal equations having hundreds of thousands of unknowns on a computer having as little as 120 Kilo Bytes of RAM.

Choleski Decomposition:

A set of normal equations may be expressed in matrix notation as:

$$N \cdot \hat{X} \quad = \quad B$$

where,

$\mathbf{N}$ represents the normal equation matrix;
$\hat{X}$ represents the solution vector to be estimated;
B represents the associated constant vector.

Since, the normal equation matrix ***N*** arising from control networks is, and always will be, positive definite symmetric, it may be decomposed into a product of two triangular matrices one of which is the transpose of the other, that is:

$$N = T \cdot T^t \,.$$

If ***N*** is banded and bordered, then ***T*** and its transpose will also be banded and bordered; with exactly the same boundary limits. A practical example will best show how the triangular matrices, ***T*** and its transpose are computed.

$$\begin{bmatrix} n_{1,1} & n_{1,2} & n_{1,3} & 0 & 0 & n_{1,6} \\ n_{1,2} & n_{2,2} & n_{2,3} & n_{2,4} & 0 & n_{2,6} \\ n_{1,3} & n_{2,3} & n_{3,3} & n_{3,4} & n_{3,5} & n_{3,6} \\ 0 & n_{2,4} & n_{34} & n_{4,4} & n_{4,5} & n_{4,6} \\ 0 & 0 & n_{3,5} & n_{4,5} & n_{5,5} & n_{5,6} \\ n_{1,6} & n_{2,6} & n_{3,6} & n_{4,6} & n_{5,6} & n_{6,6} \end{bmatrix} = \begin{bmatrix} t_{1,1} & & & & & \\ t_{1,2} & t_{2,2} & & & & \\ t_{1,3} & t_{2,3} & t_{3,3} & & & \\ 0 & t_{2,4} & t_{3,4} & t_{4,4} & & \\ 0 & 0 & t_{3,5} & t_{4,5} & t_{5,5} & \\ t_{1,6} & t_{2,6} & t_{3,6} & t_{4,6} & t_{5,6} & t_{6,6} \end{bmatrix}$$

$$\times \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & 0 & 0 & t_{1,6} \\ & t_{2,2} & t_{2,3} & t_{2,4} & 0 & t_{2,6} \\ & & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ & & & t_{4,4} & t_{4,5} & t_{4,6} \\ & & & & t_{5,5} & t_{5,6} \\ & & & & & t_{6,6} \end{bmatrix}$$

By multiplying ***T*** by its transpose to get ***N*** we see that,

$$n_{1,1} = t_{1,1} \cdot t_{1,1} \qquad t_{1,1} = \pm\sqrt{n_{1,1}}$$

$$n_{1,2} = t_{1,1} \cdot t_{1,2} \qquad t_{1,2} = n_{1,2} / t_{1,1}$$

$$n_{1,3} = t_{1,1} \cdot t_{1,3} \qquad t_{1,3} = n_{1,3} / t_{1,1}$$

$$n_{1,6} = t_{1,1} \cdot t_{1,6} \qquad t_{1,6} = n_{1,6} / t_{1,1}$$

$$n_{2,2} = t_{1,2} \cdot t_{1,2} + t_{2,2} \cdot t_{2,2} \qquad t_{2,2} = \pm\sqrt{\left[n_{2,2} - t_{1,2} \cdot t_{1,2}\right]}$$

$$n_{2,3} \quad = \quad t_{1,2} \cdot t_{1,3} + t_{2,2} \cdot t_{2,3} \qquad\qquad t_{2,3} \quad = \quad \left[n_{2,3} - t_{1,2} \cdot t_{1,3}\right] / \, t_{2,2}$$

$$n_{2,4} \quad = \quad t_{2,2} \cdot t_{2,4} \qquad\qquad t_{2,4} \quad = \quad n_{2,4} \, / \, t_{2,2}$$

$$n_{2,6} \quad = \quad t_{1,2} \cdot t_{1,6} + t_{2,2} \cdot t_{2,6} \qquad\qquad t_{2,6} \quad = \quad \left[n_{2,6} - t_{1,2} \cdot t_{1,6}\right] / \, t_{2,2}$$

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$$n_{6,6} \quad = \quad t_{1,6} \cdot t_{1,6} + t_{2,6} \cdot t_{2,6} + t_{3,6} \cdot t_{3,6} + t_{4,6} \cdot t_{4,6} + t_{5,6} \cdot t_{5,6} + t_{6,6} \cdot t_{6,6}$$

$$t_{6,6} \quad = \quad \pm \sqrt{\left[n_{6,6} - t_{1,6}^2 - t_{2,6}^2 - t_{3,6}^2 - t_{4,6}^2 - t_{5,6}^2\right]}$$

Since the triangular matrices are also banded and bordered, respective to the band and border of the normal equation matrix, one need not compute any coefficients outside of these areas. Since the normal equation matrix is symmetric all that is required to be stored is the upper triangular part. The Choleski decomposition process allows the upper triangular part of ***N*** to be directly substituted by transpose of ***T***, as the process continues, without any extra data storage required.

Solution of the Vector of Unknowns by Choleski:

The solution vector may be computed by means of a forward substitution through the lower triangular matrix and then a back substitution through the upper triangular matrix.

$$N \cdot \hat{X} \quad = \quad B$$

$$T \cdot T^t \cdot \hat{X} \quad = \quad B$$

$$T^t \cdot \hat{X} \quad = \quad T^{-1} \cdot B \quad = \quad D$$

The inverse of ***T*** does not have to be computed as it is possible to divide a vector by a triangular matrix.

$$T \cdot D \quad = \quad B$$

$$D_1 \quad = \quad B_1 \,/\, T_{1,1}$$

$$D_i \quad = \left[B_i \quad - \quad \sum_{k=1}^{i-1} T_{i,k} \cdot D_k \right] \ / \, T_{i,i} \qquad i = 2, \quad 3, \quad 4, \quad \ldots, \quad n$$

Now we can perform a back substitution to compute the vector of unknowns.

$$T^t \cdot \hat{X} \quad = \quad D$$

$$\hat{X}_n \quad = \quad D_n \,/\, T_{n,n}$$

$$\hat{X}_i \quad = \quad \left[D_i \quad - \quad \sum_{k=i+1}^{n} T_{i,k} \cdot \hat{X}_k \right] \ / \, T_{i,i} \qquad i \quad = \quad n-1, \quad n-2, \quad \ldots, \quad 1$$

A small example will best show how the vector of unknowns is solved.

Forward Solution for the Vector of Unknowns:

$$\begin{bmatrix} t_{1,1} & & & & & \\ t_{1,2} & t_{2,2} & & & & \\ t_{1,3} & t_{2,3} & t_{3,3} & & & \\ 0 & t_{2,4} & t_{3,4} & t_{4,4} & & \\ 0 & 0 & t_{3,5} & t_{4,5} & t_{5,5} & \\ t_{1,6} & t_{2,6} & t_{3,6} & t_{4,6} & t_{5,6} & t_{6,6} \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix}$$

$$t_{1,1} \cdot d_1 \quad = b_1$$

$$t_{1,2} \cdot d_1 + t_{2,2} \cdot d_2 \quad = b_2$$

$$t_{1,3} \cdot d_1 + t_{2,3} \cdot d_2 + t_{3,3} \cdot d_3 \quad = b_3$$

$$t_{2,4} \cdot d_2 + t_{3,4} \cdot d_3 + t_{4,4} \cdot d_4 \quad = b_4$$

$$t_{3,5} \cdot d_3 + t_{4,5} \cdot d_4 + t_{5,5} \cdot d_5 \quad = b_5$$

$$t_{1,6} \cdot d_1 + t_{2,6} \cdot d_2 + t_{3,6} \cdot d_3 + t_{4,6} \cdot d_4 + t_{5,6} \cdot d_5 + t_{6,6} \cdot d_6 \quad = b_6$$

$$d_1 \quad = \quad b_1 \,/\, t_{1,1}$$

$$d_2 \quad = \quad \left[b_2 - t_{1,2} \cdot d_1 \right] / \, t_{2,2}$$

$$d_3 \quad = \quad \left[b_3 - t_{1,3} \cdot d_1 - t_{2,3} \cdot d_2 \right] / \, t_{3,3}$$

$$d_4 \quad = \quad \left[b_4 - t_{2,4} \cdot d_2 - t_{3,4} \cdot d_3 \right] / \, t_{4,4}$$

$$d_5 \quad = \quad \left[b_5 - t_{3,5} \cdot d_3 - t_{4,5} \cdot d_4 \right] / \, t_{5,5}$$

$$d_6 \quad = \quad \left[b_4 - t_{1,6} \cdot d_1 - t_{2,6} \cdot d_2 - t_{3,6} \cdot d_3 - t_{4,6} \cdot d_4 - t_{5,6} \cdot d_5 \right] / \, t_{6,6}$$

Back Solution for Vector of Unknowns:

$$\begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & 0 & 0 & t_{1,6} \\ & t_{2,2} & t_{2,3} & t_{2,4} & 0 & t_{2,6} \\ & & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ & & & t_{4,4} & t_{4,5} & t_{4,6} \\ & & & & t_{5,5} & t_{5,6} \\ & & & & & t_{6,6} \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{x}_5 \\ \hat{x}_6 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}$$

$$t_{1,1} \cdot \hat{x}_1 + t_{1,2} \cdot \hat{x}_2 + t_{1,3} \cdot \hat{x}_3 + t_{1,6} \cdot \hat{x}_6 \quad = \quad d_1$$

$$t_{2,2} \cdot \hat{x}_2 + t_{2,3} \cdot \hat{x}_3 + t_{2,4} \cdot \hat{x}_4 + t_{2,6} \cdot \hat{x}_6 \quad = \quad d_2$$

$$t_{3,3} \cdot \hat{x}_3 + t_{3,4} \cdot \hat{x}_4 + t_{3,5} \cdot \hat{x}_5 + t_{3,6} \cdot \hat{x}_6 \quad = \quad d_3$$

$$t_{4,4} \cdot \hat{x}_4 + t_{4,5} \cdot \hat{x}_5 + t_{4,6} \cdot \hat{x}_6 \quad = \quad d_4$$

$$t_{5,5} \cdot \hat{x}_5 + t_{5,6} \cdot \hat{x}_6 \quad = \quad d_5$$

$$t_{6,6} \cdot \hat{x}_6 \quad = \quad d_6$$

$$\hat{x}_6 \quad = \quad d_6 \,/\, t_{6,6}$$

$$\hat{x}_5 \quad = \quad \left[d_5 - t_{5,6} \cdot \hat{x}_6\right] / \, t_{5,5}$$

$$\hat{x}_4 \quad = \quad \left[d_4 - t_{4,6} \cdot \hat{x}_6 - t_{4,5} \cdot \hat{x}_5\right] / \, t_{4,4}$$

$$\hat{x}_3 \quad = \quad \left[d_3 - t_{3,6} \cdot \hat{x}_6 - t_{3,5} \cdot \hat{x}_5 - t_{3,4} \cdot \hat{x}_4\right] / \, t_{3,3}$$

$$\hat{x}_2 \quad = \quad \left[d_2 - t_{2,6} \cdot \hat{x}_6 - t_{2,4} \cdot \hat{x}_4 - t_{2,3} \cdot \hat{x}_3\right] / \, t_{2,2}$$

$$\hat{x}_1 \quad = \quad \left[d_1 - t_{1,6} \cdot \hat{x}_6 - t_{1,3} \cdot \hat{x}_3 - t_{1,2} \cdot \hat{x}_2\right] / \, t_{1,1}$$

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