Mathematical Model for an Observed Azimuth
on a Conformal Mapping Plane

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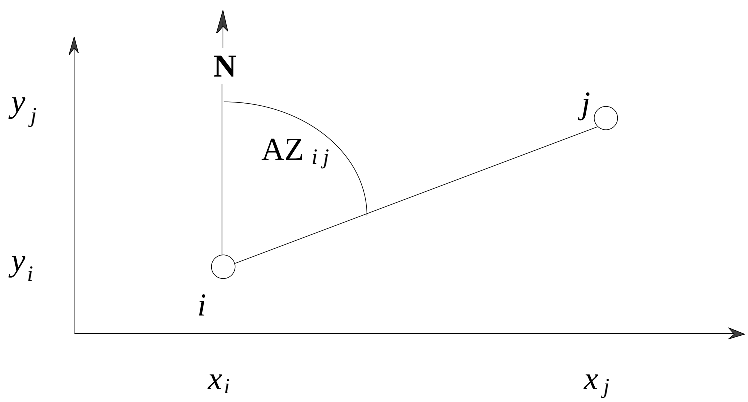
The mathematical model for an azimuth.



From basic trigonometry we have:

$$AZ_{ij} \quad = \quad \alpha_{ij} \quad = \quad \tan^{-1}\left\{ \frac{x_j - x_i}{y_j - y_i} \right\}$$

a non-linear equation. Non-linear equations are not easy to model in a least squares adjustment, therefore, we shall substitute the rigorous mathematical model with the first terms (linear terms) of the Taylor's series expansion of the mathematical model and iterate the solution to convergence. Taylor's series approximation requires a set of approximate values for the unknowns. We shall use the tilde (~) sign to denote approximate values. The set of approximate values must be calculated before a least squares adjustment is initiated.

$$F\left(\tilde{\Delta}\right) \quad = \quad \tilde{\alpha}_{ij} \quad = \quad \tan^{-1}\left\{ \frac{\tilde{x}_j - \tilde{x}_i}{\tilde{y}_j - \tilde{y}_i} \right\}$$

$$A \quad = \quad \left.\frac{\partial F}{\partial \Delta}\right|_{\tilde{\Delta}} \quad = \quad \left[\frac{\partial F}{\partial y_i}, \quad \frac{\partial F}{\partial x_i}, \quad \frac{\partial F}{\partial y_j}, \quad \frac{\partial F}{\partial x_j} \right]$$

$$\frac{\partial F}{\partial y_i} \;=\; 1 \div \left\{ 1 \;+\; \left\{ \frac{\tilde{x}_j - \tilde{x}_i}{\tilde{y}_j - \tilde{y}_i} \right\}^2 \right\} \;\times\; \frac{\left(-\left(\tilde{x}_j - \tilde{x}_i \right) \right) (-1)}{\left(\tilde{y}_j - \tilde{y}_i \right)^2}$$

$$\frac{\partial F}{\partial y_i} \;=\; \left(\tilde{x}_j - \tilde{x}_i \right) \;\div\; \left\{ \left\{ \frac{\left(\tilde{y}_j - \tilde{y}_i \right)^2 + \left(\tilde{x}_j - \tilde{x}_i \right)^2}{\left(\tilde{y}_j - \tilde{y}_i \right)^2} \right\} \times \left(\tilde{y}_j - \tilde{y}_i \right)^2 \right\}$$

$$\frac{\partial F}{\partial y_i} \;=\; \left(\tilde{x}_j - \tilde{x}_i \right) \;\div\; \tilde{S}_{ij}^2$$

$$\frac{\partial F}{\partial x_i} \;=\; 1 \div \left\{ 1 \;+\; \left\{ \frac{\tilde{x}_j - \tilde{x}_i}{\tilde{y}_j - \tilde{y}_i} \right\}^2 \right\} \;\times\; \frac{(-1)}{\left(\tilde{y}_j - \tilde{y}_i \right)}$$

$$\frac{\partial F}{\partial x_i} \;=\; (-1) \;\div\; \left\{ \left\{ \frac{\left(\tilde{y}_j - \tilde{y}_i \right)^2 + \left(\tilde{x}_j - \tilde{x}_i \right)^2}{\left(\tilde{y}_j - \tilde{y}_i \right)^2} \right\} \times \left(\tilde{y}_j - \tilde{y}_i \right) \right\}$$

$$\frac{\partial F}{\partial x_i} \;=\; -\left(\tilde{y}_j - \tilde{y}_i \right) \;\div\; \tilde{S}_{ij}^2$$

Similarly,

$$\frac{\partial F}{\partial y_j} \;=\; -\left(\tilde{x}_j - \tilde{x}_i \right) \;\div\; \tilde{S}_{ij}^2$$

$$\frac{\partial F}{\partial x_j} \;=\; +\left(\tilde{y}_j - \tilde{y}_i \right) \;\div\; \tilde{S}_{ij}^2$$

$$A = \left[\frac{+(\tilde{x}_j - \tilde{x}_i)}{\tilde{S}_{ij}^{\,2}}, \quad \frac{-(\tilde{y}_j - \tilde{y}_i)}{\tilde{S}_{ij}^{\,2}}, \quad \frac{-(\tilde{x}_j - \tilde{x}_i)}{\tilde{S}_{ij}^{\,2}}, \quad \frac{+(\tilde{y}_j - \tilde{y}_i)}{\tilde{S}_{ij}^{\,2}} \right]$$

A is known as the design matrix for the unknown parameters.

$$\hat{\Delta} = \begin{bmatrix} dy_i \\ dx_i \\ dy_j \\ dx_j \end{bmatrix}$$

$$W = \tilde{\alpha}_{ij} - \alpha_{ij} = F(\tilde{\Delta}) - \alpha \quad (observed)$$

The design matrix for the observations is:

$$B = \left.\frac{\partial F}{\partial L}\right|_L = \begin{bmatrix} -1 & 0 & . & . & . & . & 0 & 0 \\ 0 & -1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & -1 & 0 \\ 0 & 0 & . & . & . & . & 0 & -1 \end{bmatrix}$$

a negative Identity matrix, therefore, a least squares adjustment of azimuths uses the parametric case.