Elimination of Nuisance Unknowns

Dr. Peter A. Steeves, P.Eng.
Geodetic Software Systems

Sometimes, only a partial solution of the least squares estimate solution vector is required; that is, only a certain few of the unknowns need to be solved. This chapter derives matrix algebra expressions that will enable the elimination of the remaining so called "nuisance unknowns" and gives particular attention to the elimination of the orientation unknowns for the sets of observed directions.

Derivation of the Elimination of Nuisance Unknowns for the General Case

The A matrix may be partitioned so that one partition consists of coefficients pertaining to the desired portion of the solution vector and the second partition consists of the remaining coefficients. It is necessary that the vector of unknowns be partitioned to be conformable for matrix manipulations with the partitioned A matrix. The general parametric least squares matrix equation,

$$\left[A^t \cdot P \cdot A\right] \;.\; \left[\hat{\Delta}\right] \;=\; -\left[A^t \cdot P \cdot W\right] \qquad \text{......... 1}$$

can be re-written as,

$$\left[A_1\,,\,A_2\right]^t \cdot P \cdot \left[A_1\,,\,A_2\right] \cdot \begin{bmatrix} \hat{\Delta}_1 \\ \hat{\Delta}_2 \end{bmatrix} \;=\; -\left[A_1\,,\,A_2\right]^t \cdot P \cdot W \qquad \text{......... 2}$$

$$\begin{bmatrix} A_1^t \\ A_2^t \end{bmatrix} \cdot P \cdot \left[A_1\,,\,A_2\right] \cdot \begin{bmatrix} \hat{\Delta}_1 \\ \hat{\Delta}_2 \end{bmatrix} \;=\; -\begin{bmatrix} A_1^t \\ A_2^t \end{bmatrix} \cdot P \cdot W \qquad \text{......... 3}$$

By matrix multiplication, we obtain the following:

$$\begin{bmatrix} \left[A_1^t \cdot P \cdot A_1\right] & \left[A_1^t \cdot P \cdot A_2\right] \\ \left[A_2^t \cdot P \cdot A_1\right] & \left[A_2^t \cdot P \cdot A_2\right] \end{bmatrix} \;\cdot\; \begin{bmatrix} \hat{\Delta}_1 \\ \hat{\Delta}_2 \end{bmatrix} \;=\; -\begin{bmatrix} A_1^t \cdot P \cdot W \\ A_2^t \cdot P \cdot W \end{bmatrix} \qquad \text{......... 4}$$

$$\left[A_1^t \cdot P \cdot A_1\right] \cdot \hat{\Delta}_1 \quad + \quad \left[A_1^t \cdot P \cdot A_2\right] \cdot \hat{\Delta}_2 \quad = \quad -\left[A_1^t \cdot P \cdot W\right] \qquad \text{..........} \; 5$$

$$\left[A_2^t \cdot P \cdot A_1\right] \cdot \hat{\Delta}_1 \quad + \quad \left[A_2^t \cdot P \cdot A_2\right] \cdot \hat{\Delta}_2 \quad = \quad -\left[A_2^t \cdot P \cdot W\right] \qquad \text{..........} \; 6$$

From equation 6, we may write,

$$\hat{\Delta}_2 \quad = \quad -\left[A_2^t \cdot P \cdot A_2\right]^{-1} \cdot \left[A_2^t \cdot P \cdot W \quad + \quad A_2^t \cdot P \cdot A_1 \cdot \hat{\Delta}_1\right] \qquad \text{..........} \; 7$$

and by substituting $\hat{\Delta}_2$ into equation 5 we get,

$$A_1^t P A_1 \cdot \hat{\Delta}_1 - A_1^t P A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot \left[A_2^t PW + A_2^t P A_1 \cdot \hat{\Delta}_1\right] = -A_1^t PW \qquad \text{..........} \; 8$$

$$A_1^t P A_1 \cdot \hat{\Delta}_1 - A_1^t P A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P A_1 \cdot \hat{\Delta}_1 \quad = \quad -A_1^t PW +$$

$$A_1^t P A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t PW \qquad \text{..........} \; 9$$

$$\left[A_1^t P A_1 - A_1^t P A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P A_1\right] \cdot \hat{\Delta}_1 \quad = \quad -A_1^t PW \quad +$$

$$A_1^t P A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t PW \qquad \text{..........} \; 10$$

$$A_1^t P \cdot \left[A_1 - A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P A_1\right] \cdot \hat{\Delta}_1 \quad =$$

$$-A_1^t P \cdot \left[W - A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t PW\right] \qquad \text{..........} \; 11$$

$$A_1^t P \cdot \left[I - A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P\right] \cdot A_1 \hat{\Delta}_1 \quad =$$

$$-A_1^t P \cdot \left[I - A_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P\right] \cdot W \qquad \text{..........} \; 12$$

$$A_1^t \cdot \left[P - PA_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P\right] \cdot A_1 \hat{\Delta}_1 \quad =$$

$$-A_1^t \cdot \left[P - PA_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P\right] \cdot W \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 13$$

Let U = $\left[P - PA_2 \cdot \left[A_2^t P A_2\right]^{-1} \cdot A_2^t P\right]$ 14

then,

$$\left[A_1^t \cdot U \cdot A_1\right] \cdot \hat{\Delta}_1 \quad = -\left[A_1^t \cdot U \cdot W\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 15$$

$$\hat{\Delta}_1 \quad = \quad -\left[A_1^t \cdot U \cdot A_1\right]^{-1} \cdot \left[A_1^t \cdot U \cdot W\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 16$$

Equation 16 is similar to,

$$\hat{\Delta} \quad = \quad -\left[A^t \cdot P \cdot A\right]^{-1} \cdot \left[A^t \cdot P \cdot W\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 17$$

hence, the matrix U may be considered a transformation of the weight matrix P, which allows the solution of $\hat{\Delta}_1$, without solving for $\hat{\Delta}_2$.

Elimination of Orientation Unknowns

In a horizontal control adjustment, the orientation unknowns are considered as nuisance unknowns. In this particular case the matrix U is extremely easy to compute, the method will be shown below. In the chapter on creating the normal equations, it was shown how to create the normal equation matrix by operating on the observation equations from one station at a time; the same procedure will be used here. If there are *n* directions taken at a station *i*, then the $[A_2]$ partition (corresponding to the orientation unknowns) has the following characteristics.

$$\underset{n\ 1}{A_2} = \begin{bmatrix} -1 \\ -1 \\ \cdots \\ \cdots \\ -1 \end{bmatrix}$$

then,

$$A_2^t \cdot P \cdot A_2 = \begin{bmatrix} -1 & -1 & \cdots & \cdots & -1 \end{bmatrix} \cdot \begin{bmatrix} P_1 & & & & \\ & P_2 & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & P_n \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ \cdots \\ \cdots \\ -1 \end{bmatrix}$$

$$A_2^t \cdot P \cdot A_2 = \sum_{k=1}^{n} P_k \qquad \text{(a scalar)}$$

and,

$$\left[A_2^t \cdot P \cdot A_2\right]^{-1} = 1.0 \div \sum_{k=1}^{n} P_k$$

$$P \cdot A_2 = \begin{bmatrix} P_1 & & & & \\ & P_2 & & & \\ & & \dots & & \\ & & & \dots & \\ & & & & P_n \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ \dots \\ \dots \\ -1 \end{bmatrix} = \begin{bmatrix} -P_1 \\ -P_2 \\ \\ \\ -P_n \end{bmatrix}$$

$$\left[P \cdot A_2\right] \cdot \left[A_2^t \cdot P \cdot A_2\right]^{-1} = 1.0 \div \sum_{k=1}^{n} P_k \cdot \begin{bmatrix} -P_1 \\ -P_2 \\ \dots \\ \dots \\ -P_n \end{bmatrix}$$

$$\left[A_2^t \cdot P\right] = \begin{bmatrix} -1 & -1 & \dots & \dots & -1 \end{bmatrix} \cdot \begin{bmatrix} P_1 & & & & \\ & P_2 & & & \\ & & \dots & & \\ & & & \dots & \\ & & & & P_n \end{bmatrix} = \begin{bmatrix} -P_1 & -P_2 & \dots & \dots & -P_n \end{bmatrix}$$

$$P \cdot A_2 \cdot \left[A_2^t \cdot P \cdot A_2\right]^{-1} \cdot A_2^t \cdot P = 1.0 \div \sum_{k=1}^{n} P_k \cdot \begin{bmatrix} -P_1 \\ -P_2 \\ \dots \\ \dots \\ -P_n \end{bmatrix} \cdot \begin{bmatrix} -P_1 & -P_2 & \dots & \dots & -P_n \end{bmatrix}$$

$$= 1.0 \div \sum_{k=1}^{n} P_k \cdot \begin{bmatrix} P_1^2 & P_1 P_2 & P_1 P_3 & \dots & \dots & P_1 P_n \\ P_2 P_1 & P_2^2 & P_2 P_3 & \dots & \dots & P_2 P_n \\ P_3 P_1 & P_3 P_2 & P_3^2 & \dots & \dots & P_3 P_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ P_n P_1 & P_n P_2 & P_n P_3 & \dots & \dots & P_n^2 \end{bmatrix}$$

$$U = \left[P - P \cdot A_2 \cdot \left[A_2^t \cdot P \cdot A_2\right]^{-1} \cdot A_2^t \cdot P\right]$$

$$U = \begin{bmatrix} P_1 + P_1^2 \ddot{Z} & P_1 P_2 \ddot{Z} & P_1 P_3 \ddot{Z} & \dots & \dots & P_1 P_n \ddot{Z} \\ P_2 P_1 \ddot{Z} & P_2 + P_2^2 \ddot{Z} & P_2 P_3 \ddot{Z} & \dots & \dots & P_2 P_n \ddot{Z} \\ P_3 P_1 \ddot{Z} & P_3 P_2 \ddot{Z} & P_3 + P_3^2 \ddot{Z} & \dots & \dots & P_3 P_n \ddot{Z} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ P_n P_1 \ddot{Z} & P_n P_2 \ddot{Z} & P_n P_3 \ddot{Z} & \dots & \dots & P_n + P_n^2 \ddot{Z} \end{bmatrix}$$

where,

$$\ddot{Z} = -1.0 \div \sum_{k=1}^{n} P_k$$