Sequential Formation of Normal Equations

Dr. Peter A. Steeves, P.Eng.
Geodetic Software Systems

Let the normal equation matrix be denoted by, $N = A^t \cdot A$. By allowing A to be partitioned as follows,

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix}$$

we may write,

$$N = A^t \cdot A = A_1^t \cdot A_1 + A_2^t \cdot A_2 + A_3^t \cdot A_3 + \ldots\ldots\ldots A_m^t \cdot A_m$$

It is evident that the normal equation matrix (N) and constant vector $\left(A^t \cdot W\right)$ may, be formed sequentially, by operating with any number of rows of the design matrix at a time. Assume the normal equation matrix has order equal to eight and the first row of the design matrix is:

$$\underset{1\ \ 8}{A_1} = \begin{bmatrix} 2 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}$$

The additive contribution to the normal equation matrix from $A_1^t \cdot A_1$, is:

$$\begin{bmatrix}2\\2\\0\\0\\2\\2\\0\\1\end{bmatrix} \cdot \begin{bmatrix}2 & 2 & 0 & 0 & 2 & 2 & 0 & 1\end{bmatrix} = \begin{bmatrix}4 & 4 & 0 & 0 & 4 & 4 & 0 & 2\\4 & 4 & 0 & 0 & 4 & 4 & 0 & 2\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\4 & 4 & 0 & 0 & 4 & 4 & 0 & 2\\4 & 4 & 0 & 0 & 4 & 4 & 0 & 2\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\2 & 2 & 0 & 0 & 2 & 2 & 0 & 1\end{bmatrix}$$

If we allow each row of the design matrix to be compacted (eliminate all zero entries) and allow the respective column numbers of the non-zero coefficients to follow the non-zero coefficients on the data file, row by row, the following is possible:

Call the first row,

$$\begin{bmatrix} \bar{A}_1 \\ _{1 \quad 5} \end{bmatrix} \quad \begin{bmatrix} C_1 \\ _{1 \quad 5} \end{bmatrix} = \begin{bmatrix}2 & 2 & 2 & 2 & 1 & \| & 1 & 2 & 5 & 6 & 8\end{bmatrix}$$

$\bar{A}_1$ = the compacted first row of A.

C_1 = the column numbers of the respective coefficients of $\bar{A}_1$.

then,

$$\bar{A}_1^t \cdot \bar{A}_1 = \begin{bmatrix}2\\2\\2\\2\\1\end{bmatrix} \cdot \begin{bmatrix}2 & 2 & 2 & 2 & 1\end{bmatrix} = \begin{bmatrix}4 & 4 & 4 & 4 & 2\\4 & 4 & 4 & 4 & 2\\4 & 4 & 4 & 4 & 2\\4 & 4 & 4 & 4 & 2\\2 & 2 & 2 & 2 & 1\end{bmatrix}$$

The compacted contribution to the normal equation matrix can be added to the normal equation matrix, by using the column numbers as indices for the addition. In this particular example, the indices are (1, 2, 5, 6, 8). The indices can be used to instruct the computer to make sums to the (1, 2, 5, 6, 8) columns of each of the (1, 2, 5, 6, 8) rows of the normal equation matrix.

The same procedure can be used for the computation of the constant and residual vectors.

This simple procedure makes it possible to process thousands of rows of the design matrix (A) per second. In actual fact, there is no reason to store the design matrix at all. Our program simply computes each row of the design matrix and the accumulations to the normal equation matrix directly from rows or small groups of rows of the raw data. The design matrix is never stored.